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Nonlinear effects in steady supersonic dissipative gasdynamics. Part 2. Three-dimensional axisymmetric flow

By T. H. CHONG AND L. SIROVICH
Division of Applied Mathematics and Centre for Fluid Mechanics,
Brown University, Providence, R.I.

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Steady, supersonic, dissipative, three-dimensional, axisymmetric flow is considered. A system of Burgers-type equations is shown to govern the flow field. In inviscid regions the Whitham theory gives the limiting form. Dissipative effects ultimately engulf the inviscid zone and at sufficiently large distances from the body the flow is governed by linear dissipative theory. The flow field is divided into zones based on the presence or absence of nonlinearity and dissipation. Estimates and criteria which describe the extent of these zones are given.

1. Introduction

The results of the present treatment of supersonic axisymmetric flow differ in a number of respects from the results of our investigation of two-dimensional flow (Chong & Sirovich 1971, hereafter referred to as I). Two of the more striking differences are (i) that at large distances the picture of distinct dissipative shock and inviscid zones fails, dissipation completely invading the wave region; and (ii) that at larger distances linear dissipative theory is found to govern the flow.

In our investigation we first demonstrate the breakdown of linear theory. The multiple scales method (Cole 1968, p. 79) is then used and results in a system of governing equations each resembling the Burgers equation (Burgers 1948). In fact a transformation reduces the basic equation to a Burgers equation containing a spatially dependent 'diffusivity'. In those regions where inviscid effects can be separated, our results reduce to those of the Whitham theory (Whitham 1950, 1952, 1956). However, at sufficiently large distances from the body no inviscid region exists.

Dissipation has already been considered in the transonic studies of Ryzhov (1965), Ryzhov & Shefter (1964) and Szaniawski (1968). In these studies similarity solutions appear to be physically relevant whereas this is demonstrated not to be so in our case. The most important earlier treatment is due to Lighthill (1956, especially §9), who, using a different viewpoint and formulation, also derived a Burgers equation with a spatially dependent diffusivity. (The diffusivity grows with altitude and this lies at the root of the novel results mentioned in the first paragraph of this section.)

2. Governing equations

Although we are primarily interested in three-dimensional flows, the two-dimensional case is also included. Also, since the notation and normalization are the same as in I, the reader is asked to look there for such details.

We recall that $\mathbf{v} = (\rho, u, v, T)$ refers to perturbation quantities. The upstream supersonic velocity is taken to be in the x direction and the direction perpendicular to this is denoted by y; u and v denote the velocity perturbations in these directions, respectively, while ρ and T represent the density and temperature perturbations, respectively. The governing equations are written as

$$\left[\mathbf{A}\frac{\partial}{\partial x}+\mathbf{B}\frac{\partial}{\partial y}+\frac{n-2}{y}\mathbf{C}\right]\mathbf{v} = \mathbf{X}(\mathbf{v})+\mathbf{Y}(\mathbf{v})+\mathbf{Z}(\mathbf{v}), \tag{1}$$

where $n = 2$ or 3 represents the number of dimensions. The matrices $\mathbf{A}$ and $\mathbf{B}$ are given on page 166 of I and $(C)_{ij} = \delta_{i1}\delta_{j3}+\chi\,\delta_{i4}\delta_{j3}$. The vector function $\mathbf{X}$ represents quadratic inviscid terms, $\mathbf{Y}$ the linear dissipative terms and $\mathbf{Z}$ the remaining terms.

3. Breakdown of linear theory

Inviscid theory

Dissipation will be neglected for the moment. For flow past a thin or slender body of thickness ratio ϵ we can formally expand $\mathbf{v}$:

$$\mathbf{v} = \epsilon\mathbf{v}_1+\epsilon^2\mathbf{v}_2+\dots. \tag{2}$$

All lengths are now regarded as normalized with respect to the body length L. From (1) to lowest order the flow is governed by

$$\left[\mathbf{A}\frac{\partial}{\partial x}+\mathbf{B}\frac{\partial}{\partial y}+\frac{n-2}{y}\mathbf{C}\right]\mathbf{v}_1 = 0. \tag{3}$$

As is well known, (3) may be reduced to the wave equation. Since we are interested in regions well away from the object we solve by a procedure which proves useful in systematically generating higher order terms.

The matrix $\mathbf{A}$ is positive definite in supersonic flows and from this it follows that there exists a complete set of real bi-orthogonal eigenvectors $\{\mathbf{l}^i\}$ and $\{\boldsymbol{\alpha}^i\}$ and real eigenvalues $\{\lambda^i\}$ such that

$$\left.\begin{aligned}\lambda^i\mathbf{A}\mathbf{l}^i = \mathbf{B}\mathbf{l}^i = \lambda^i\boldsymbol{\alpha}^i,\\ \boldsymbol{\alpha}^i.\mathbf{l}^j = 0, \quad i \neq j,\end{aligned}\right\}\quad (i = 1,2,3,4). \tag{4}$$

The λ^i, $\mathbf{l}^i$, and $\boldsymbol{\alpha}^i$ are given by equations (26)–(28) of I.

Without loss of generality we can decompose $\mathbf{v}_1$ into the four 'modes':

$$\mathbf{v}_1 = \mathbf{l}^1 g_1(x,y)+\mathbf{l}^2 g_2(x,y)+\mathbf{l}^3 g_3(x,y)+\mathbf{l}^4 g_4(x,y), \tag{5}$$

with $g_i = \boldsymbol{\alpha}^i.\mathbf{v}_1/\boldsymbol{\alpha}^i.\mathbf{l}^i$. On substituting (5) into (3) and multiplying from the left by $\mathbf{l}^i$, we find that for $i = 1$ and 2 (corresponding to the wake) the equation decouples completely into

$$\partial g_1/\partial x = 0, \quad \partial g_2/\partial x = 0. \tag{6}$$

This leads to the inviscid form of the wake, so that in the wake

$$\mathbf{v}_1 \sim \mathbf{l}^1 g_1(y) + \mathbf{l}^2 g_2(y). \tag{7}$$

For $i = 3$ and 4 we get two coupled equations in g_3 and g_4, which in fact may be reduced to the wave equation. Instead of following this approach, we can seek the solution outside the wake by writing

$$\mathbf{v}_1 \sim \sum_{j=1} y^{-p_j} \mathbf{f}_j(x, y), \quad p_{j+1} > p_j, \tag{8}$$

and letting $y \to \infty$. In addition we require that this sum be orthogonal to $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2$ By inserting (8) into (3) we obtain

$$\left(\mathbf{A}\frac{\partial}{\partial x} + \mathbf{B}\frac{\partial}{\partial y}\right)\mathbf{f}_1 = 0,$$

which after the orthogonality condition has been imposed gives

$$\mathbf{f}_1 = \mathbf{l}^3 g(\tau) + \mathbf{l}^4 \hat{g}(\hat{\tau}), \tag{9}$$

where $\tau = y - \lambda^3 x$ and $\hat{\tau} = y - \lambda^4 x$. Since a wave travelling upstream is physically impossible, we have $\hat{g} = 0$ and

$$\mathbf{f}_1 = \mathbf{l}^3 g(\tau). \tag{10}$$

(For convenience, we consider the Mach region only in the upper half-plane in two dimensions.) It is clear that each $\mathbf{f}_i$ is in fact a function of τ only, and we now put $\mathbf{f}_i = \mathbf{f}_i(\tau)$. As we shall see, this automatically meets the orthogonality condition also. The equation to the next order is then

$$[-\lambda^3 \mathbf{A} + \mathbf{B}]\,\mathbf{f}_2' - [p_1 \mathbf{B} - (n-2)\,\mathbf{C}]\,\mathbf{f}_1 = 0. \tag{11}$$

(To obtain this we also need the matching requirement $p_{j+1} = p_j + 1$.) Multiplying (11) from the left by $\mathbf{l}^3$ yields the 'Fredholm' condition,

$$\mathbf{l}^3 . [p_1 \mathbf{B} - (n-2)\,\mathbf{C}]\,\mathbf{l}^3 g = 0,$$

which gives $p_1 = \frac{1}{2}(n-2)$. Solving (11) we find

$$\mathbf{f}_2 = \tfrac{1}{4}\mathbf{l}^4 g_{-1}(\tau) + \mathbf{l}^3 h(\tau), \tag{12}$$

where

$$g_{-1} = \int^{\tau} g(\tau')\, d\tau'$$

is an indefinite integral of g, and h is an unknown function. The latter is determined by proceeding to the next order, at which (3) has the form

$$[-\lambda^3 \mathbf{A} + \mathbf{B}]\,\mathbf{f}_3' - [p_2 \mathbf{B} - (n-2)\,\mathbf{C}]\,\mathbf{f}_2 = 0.$$

Multiplying this from the left by $\mathbf{l}^3$ finally gives

$$h(\tau) = \tfrac{1}{8} g_{-1}(\tau). \tag{13}$$

This process may be continued and the expansion for $\mathbf{v}_1$ found to all orders.

For $n = 2$ we have $p_1 = 0$ and the series for $\mathbf{v}_1$ terminates, so that

$$\mathbf{v}_1 = \mathbf{l}^3 g(\tau) \tag{14}$$

is exact. For $n = 3$, the leading term is given by

$$\mathbf{v}_1 = y^{-\frac{1}{2}}\mathbf{l}^3 g(\tau) + O(y^{-\frac{3}{2}}). \tag{15}$$

We now regard the expansion of $\mathbf{v}_1$ as known and return to the solution of the inviscid form of (1) under the expansion (2). The second-order equation is

$$\left[\mathbf{A}\frac{\partial}{\partial x} + \mathbf{B}\frac{\partial}{\partial y} + \frac{n-2}{y}\mathbf{C}\right]\mathbf{v}_2 = \mathbf{X}(\mathbf{v}_1), \tag{16}$$

in which $\mathbf{X}$ is quadratic. The solution $\mathbf{v}_2$ consists of a homogeneous solution of the same form as $\mathbf{v}_1$ and a particular solution. In the wave region defined by τ fixed and y large, we have $\mathbf{v}_1 \sim y^{-\frac{1}{2}(n-2)}\mathbf{l}^3 g(\tau)$, so that $\mathbf{X} = y^{-(n-2)}X_1$ with $X_1 = O(1)$.

To find a particular solution of (16) we first decompose $\mathbf{v}_2$ into the form

$$\mathbf{v}_2 = \mathbf{l}^1\hat{b}_1(x,y) + \mathbf{l}^2\hat{b}_2(x,y) + \mathbf{l}^3\hat{b}_3(x,y) + \mathbf{l}^4\hat{b}_4(x,y). \tag{17}$$

In analogy with (8) we write

$$\hat{b}_i \sim b_i(\tau)/y^{q_i} \quad (i = 1, 2, 3, 4).$$

Substituting into (16) and multiplying on the left by $\mathbf{l}^i$ we obtain

$$-\lambda^i(\mathbf{l}^i.\mathbf{A}\mathbf{l}^i)\,y^{-q_i}b_i' = y^{-(n-2)}\mathbf{l}^i.\mathbf{X}_1 \quad \text{for} \quad i = 1, 2, 4, \tag{18}$$

and $$(n-2)\,\mathbf{l}^3.\mathbf{C}(\mathbf{l}^3 y^{-q_3}b_3 + \mathbf{l}^4 y^{-q_4}b_4) - (\mathbf{l}^3.\mathbf{B}\mathbf{l}^3)\,q_3 y^{-q_3+1}b_3 = y^{-(n-2)}\mathbf{l}^3.\mathbf{X}_1. \tag{19}$$

The matching condition then gives $q_1 = q_2 = q_4 = n-2$ and $q_3 = n-3$. Hence the b_i are all determined by (18) and (19). In particular

$$b_3 = \frac{U}{\gamma^2(4-n)\,(M^2-1)^{\frac{1}{2}}}\mathbf{l}^3.\mathbf{X}_1(\mathbf{v}_1),$$

which determines the leading term of $\mathbf{v}_2$. By proceeding in this manner we may systematically obtain the perturbation solutions of $\mathbf{v}$ to all orders. The calculation of second-order supersonic flow has already been considered by Van Dyke (1952).

We have shown that in the wave region, i.e. τ fixed and y large, the solution to the inviscid problem has the form

$$\hat{\mathbf{v}} = \begin{cases} \mathbf{v}_0 + \epsilon O(1) + \epsilon^2 O(y) + \dots & \text{in two dimensions,} \quad (20) \\ \mathbf{v}_0 + \epsilon O(y^{-\frac{1}{2}}) + \epsilon^2 O(1) + \dots & \text{in three dimensions,} \quad (21) \end{cases}$$

which indicates that linear inviscid theory fails when $y \geqslant O(1/\epsilon)$ in two dimensions and $y \geqslant O(1/\epsilon^2)$ in three dimensions.

We now turn to the calculation of $\mathbf{v}_2$ in the wake, i.e. for $x \to \infty$ and y fixed. In computing $\mathbf{v}_2$ in the wave region we made use of the fact that the first-order wake solution (7) vanishes outside the wake, so that in fact $\mathbf{X}(\mathbf{v}_1) \sim \mathbf{X}(\mathbf{l}^3 g(\tau)/y^{\frac{1}{2}(n-2)})$. In order to obtain $\mathbf{v}_2$ in the wake we must first examine the wave portion of $\mathbf{v}_1$ in the wake, i.e. we consider g for $x \to \infty$ with y fixed. However, this function satisfies the wave equation and its properties in two and three dimensions are well known (see e.g. Ward 1955, pp. 45ff). In two dimensions g clearly is zero in this limit and in three dimensions simple estimates shows that g is of $O(x^{-3})$ in this limit. Therefore, for the wake, we can write

$$\mathbf{X}(\mathbf{v}_1) = \mathbf{X}(\mathbf{1}^1 g_1(y) + \mathbf{1}^2 g_2(y) + (n-2)\, O(x^{-3})).$$

A direct calculation then shows that

$$\mathbf{1}^{1,2}.\mathbf{X} = \begin{cases} 0, & n = 2, \\ O(x^{-3}), & n = 3. \end{cases}$$

From this it follows that no secularity appears in the inviscid analysis of the wake. However, including dissipation alters the situation, as we shall see in the next section.

Dissipative effects

Considering the wake first, we include the dissipative term $\mathbf{Y}$ in (16). Then, multiplying from the left by $\mathbf{1}^{1,2}$, we find that

$$\mathbf{1}^{1,2}.\mathbf{A}\,\partial\mathbf{v}_2/\partial x \sim \mathbf{1}^{1,2}.\mathbf{Y}(\mathbf{v}_1).$$

Since the right-hand side is a function of y only, according to (7), we find that the particular solution for $\mathbf{v}_2$ is

$$\mathbf{v}_2 = O(x\mathbf{v}_1) \quad \text{as} \quad x \to \infty,$$

which indicates secularity.

Finally we consider the shock zone. We first approximate the leading portion of a thin sharp body by a wedge or cone, depending on the number of dimensions. The shock leaving the leading part is of a constant strength and is proportional to the angle of the wedge or cone which is of $O(\epsilon)$. According to weak-shock-wave theory, the shock thickness δ is given by

$$\delta = O(\mu/\epsilon\rho_0 a_0), \tag{22}$$

where μ is the viscosity, ρ_0 is the undisturbed density and a_0 the speed of sound.

On the other hand, a linear viscous analysis of the same problem (Chong & Sirovich 1970) yields a shock thickness δ_l given by

$$\delta_l = O((\mu y/\rho_0 a_0)^{\frac{1}{2}}).$$

Thus linear theory fails at a distance y, where δ_l becomes as large as δ or

$$y/\delta > O(1/\epsilon). \tag{23}$$

4. Nonlinear description

To overcome the difficulty encountered because of the breakdown of linear theory at large distances, we employ the method of multiple scales (Cole 1968). This method attempts to overcome the appearance of secularities by the explicit introduction of the slow variation that is suggested by non-uniform expansions such as (20) and (21). We first expand our solution in the form

$$\hat{\mathbf{v}} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + p(\epsilon)\,\mathbf{v}_2 + \ldots, \tag{24}$$

where $p(\epsilon) = o(\epsilon)$ is not yet known, and let

$$\mathbf{v}_i = \mathbf{v}_i(x_0, y_0; x_1, y_1) \quad (i = 1, 2, \ldots),$$

where x_0 and y_0 are the fast scales (formerly x and y) and x_1 and y_1 are slow scales defined by

$$x_1 = \nu(\epsilon)x, \quad y_1 = \nu(\epsilon)y, \tag{25}$$

with $\nu \to 0$ as $\epsilon \to 0$; $\nu(\epsilon)$ is to be determined. The region of interest is now specified by the condition $\tau_0 = y_0 - \lambda^i x_0$, x_1 and y_1 all fixed and $\epsilon \to 0$.

From the previous section $\mathbf{v}_1$ is known to have the form given by (7), (14) and (15), i.e. in the Mach region

$$\mathbf{v}_1 \sim \mathbf{l}^3 g(y_0 - \lambda^3 x_0, x_1, y_1)/y_0^{\frac{1}{2}(n-2)} \tag{26}$$

and in the wake

$$\mathbf{v}_1 \sim \mathbf{l}^1 g_1(y_0, x_1, y_1) + \mathbf{l}^2 g_2(y_0, x_1, y_1). \tag{27}$$

As is indicated the slow dependence on x_1 and y_1 has not yet been determined. Proceeding to the next order we obtain,

$$p\left[\mathbf{A}\frac{\partial}{\partial x_0} + \mathbf{B}\frac{\partial}{\partial y_0} + \frac{n-2}{y_0}\mathbf{C}\right]\mathbf{v}_2 = -\epsilon\nu\left[\mathbf{A}\frac{\partial}{\partial x_1} + \mathbf{B}\frac{\partial}{\partial y_1}\right]\mathbf{v}_1 + \epsilon^2\mathbf{X}(\mathbf{v}_1) + \epsilon\mathbf{Y}(\mathbf{v}_1). \tag{28}$$

Both nonlinear and dissipative terms are included since, as we have seen, both can lead to secularities. Multiplying (28) on the left by $\mathbf{l}^i$ gives

$$\left(\frac{\partial}{\partial x_0} + \lambda^i\frac{\partial}{\partial y_0}\right)\mathbf{l}^i.\mathbf{A}\mathbf{v}_2 + \frac{n-2}{y_0}\mathbf{l}^i.\mathbf{C}\mathbf{v}_2 = \frac{1}{p}\mathbf{l}^i.\left\{-\epsilon\nu\left[\mathbf{A}\frac{\partial}{\partial x_1} + \mathbf{B}\frac{\partial}{\partial y_1}\right]\mathbf{v}_1 + \epsilon^2\mathbf{X} + \epsilon\mathbf{Y}\right\}. \tag{29}$$

We use the added latitude gained by introducing 'slow variables' to force the right-hand side of (29) to vanish for $x^2 + y^2 \to \infty$. Equivalently we require

$$\lim_{\epsilon\to 0}\mathbf{l}^i.\left\{-\epsilon\nu\left[\mathbf{A}\frac{\partial}{\partial x_1} + \mathbf{B}\frac{\partial}{\partial y_1}\right]\mathbf{v}_1 + \epsilon^2\mathbf{X} + \epsilon\mathbf{Y}\right\} = 0 \quad (i = 1, 2, 3, 4). \tag{30}$$

In (30), τ_0, x_1 and y_1 are held fixed and τ_0 equals $y_0 - \lambda^3 x_0$ in the Mach region and y_0 in the wake.

Mach zone: $\tau_0 = y_0 - \lambda^3 x_0 = y_0 - x_0/(M^2-1)^{\frac{1}{2}}$

Substituting (26) into (30) results in $\nu = \epsilon^{n-1}$ and from this $p = \epsilon^n$. The dissipative term $\epsilon\mathbf{Y}$ is of $O(R^{-1}\epsilon\nu^{\frac{1}{2}(n-2)})$, where R is the Reynolds number based on body length. Since R is large this term appears to be small; however, in a shock region the shock thickness and not the body length is the appropriate scaling. If (22) is used for this purpose the dissipative term $\epsilon\mathbf{Y} = O(\epsilon^3 R\nu^{\frac{1}{2}(n-2)})$ and the nonlinear term $\epsilon^2\mathbf{X} = O(\epsilon^3 R\nu^{n-2})$. For $n = 2$ viscous and nonlinear terms are of the same order, while in three dimensions the viscous term dominates. The latter conclusion arises from the failure to consider the decay of shock strength with distance; discussion of this point is deferred to the next section. Rather than redetermining ν and hence finding the appropriate expansion in the shock regions, we attempt a uniform description, i.e. all leading terms in passing through the inviscid and dissipative zones are retained.

Returning to (30), with $i = 3$ we find

$$\frac{\partial g}{\partial y_1}+cy_1^{-\frac{1}{2}(n-2)}g\frac{\partial g}{\partial \tau_0}=\frac{1}{R}\frac{\partial^2 g}{\partial \tau_0^2}, \tag{31}$$

with
$$c=\frac{(\gamma+1)M^2}{2(M^2-1)},\quad R=\frac{2\gamma^{\frac{3}{2}}(M^2-1)^{\frac{3}{2}}\nu}{M^3\{\gamma(\zeta+\eta)+\chi^2\xi\}}$$

(see I, §4 for definitions of ζ, ξ and η). Here we have set $\tau_1 = y_1-\lambda^3 x_1$, which shows that the τ_1 dependence can be suppressed.

At this point we may regard ϵ as having been only a formally small parameter and eliminate it by setting it equal to one. This gives

$$\frac{\partial g}{\partial y}+cy^{-\frac{1}{2}(n-2)}g\frac{\partial g}{\partial \tau}=\frac{1}{R}\frac{\partial^2 g}{\partial \tau^2}. \tag{32}$$

For $n = 2$, equation (32) reduces to the Burgers equation (Burgers 1948) found in I and for which an explicit solution exists as indicated there. For $n = 3$, no general exact solution has been found, although $g = G(\tau/y^{\frac{1}{2}})$ with G such that $-\frac{1}{2}\zeta G'+cGG' = G''/R$, $\zeta = \tau/y$, is a similarity solution. We shall see that this does not correspond to the physical solution. An alternative to (32) is

$$\frac{\partial g}{\partial \eta}+cg\frac{\partial g}{\partial \tau}=\frac{y^{\frac{1}{2}(n-2)}}{R}\frac{\partial^2 g}{\partial \tau^2},\quad \eta=\frac{2}{4-n}y^{\frac{1}{2}(4-n)}. \tag{33}$$

Lighthill (1956; see equation 186) derives (33) in a different way, but does not focus attention on the form of spatially dependent 'diffusivity'. Note that our demonstration shows the diffusivity to diverge as $y\to\infty$, which implies that linear theory takes over in this limit. This is made more precise in the next section.

Inviscid limit

The inviscid limit is effectively obtained by setting $R = \infty$ in (33), resulting in the so-called Hopf equation:

$$\frac{\partial g}{\partial \eta}+cg\frac{\partial g}{\partial \tau}=0. \tag{34}$$

For flow past a finite body, the zone of disturbance created by the body is contained between the head and tail shocks and we refer to this as the wave region.

In two dimensions the wave region at large distances spreads as $O((\epsilon y)^{\frac{1}{2}})$, whereas according to our transformation, equation (33), the analogous three-dimensional spread is of $O(\epsilon^{\frac{1}{2}}y^{\frac{1}{4}})$ (Lighthill 1956; Whitham 1952).

The theory of Whitham (1950, 1952, 1956) can be shown to correspond to the inviscid limit of our analysis. Whitham represents his solution in the form

$$u=-F(\zeta)/(M^2-1)^{\frac{1}{4}}(2y)^{\frac{1}{2}(n-2)}, \tag{35}$$

where ζ is the characteristic, defined by

$$x=(M^2-1)^{\frac{1}{2}}y-\hat{k}F(\zeta)y^{\frac{1}{2}(n-2)}+\zeta,\quad \hat{k}=\frac{(\gamma+1)M^4}{2^{\frac{1}{2}}(M^2-1)^{\frac{3}{4}}}.$$

By differentiating (35) along characteristics one can easily show that (35) satisfies (34) under appropriate normalization.

Solution	Two-dimensions	Three dimensions
Linear inviscid	ϵ	$\epsilon y^{-\frac{1}{2}}$
Nonlinear inviscid: head shock (Lighthill 1956; Whitham 1952)	$\epsilon y^{-\frac{1}{2}}$	$\epsilon y^{-\frac{3}{4}}$
Linearized dissipative (Sirovich 1968; Chong & Sirovich 1970; Salathé 1969)	$\epsilon (Ry)^{-\frac{1}{2}}$	$\epsilon (Ry)^{-\frac{5}{4}}$

Table 1. Fall-off in strength of supersonic flow

Wake: $\tau_0 = y_0$

To complete the analysis of the flow we turn to the wake region and consider (30) for $\mathbf{l}^i = \mathbf{l}^1, \mathbf{l}^2$. Since

$$\mathbf{l}^1 . \mathbf{X}(\mathbf{v}_1) = \mathbf{l}^2 . \mathbf{X}(\mathbf{v}_1) \approx 0$$

the problem is linear. Furthermore, from the matching condition we have

$$\nu = O(1/R),$$

i.e. the stretching ν involves just the viscous scale. (A similar effect appears on carrying the detailed stretching in the shock layers.) The resulting equations are (after removing the scaling)

$$\frac{\partial}{\partial x} g_1 - \frac{\zeta}{U}\left(\frac{\partial^2}{\partial y^2} + \frac{n-2}{y}\frac{\partial}{\partial y}\right) g_1 = 0,$$

$$\frac{\partial}{\partial x} g_2 - \frac{\xi}{\gamma U}\left(\frac{\partial^2}{\partial y^2} + \frac{n-2}{y}\frac{\partial}{\partial y}\right) g_2 = 0.$$

Therefore in addition to linearity, we also have the decoupling of the viscous and heat conduction wakes. Also, as may be verified, the former is a vorticity and the latter an entropy wake. These wakes are treated in the linear cases (Sirovich 1968; Chong & Sirovich 1970) and further discussion is not deemed necessary.

5. Behaviour in the wave region

Table 1 contains a summary of the fall-off in strength of various solutions in the wave zone.

The first line of table 1 contains well-known results which follow from simple acoustic theory. The second line shows the effect of nonlinearity as is felt at a head shock, and the third line contains results from linearized dissipative flow. In two dimensions, dissipation and nonlinearity are equally effective and both effects must be given equal weight, a fact already manifested in the Burgers equation which describes two-dimensional supersonic flow. On the other hand, in the three-dimensional case, nonlinearity becomes less effective with distance since the dissipative mechanism becomes the strongest source of decay. From this we conclude that in three dimensions the linearized dissipative equations take over and describe the flow at large distances. Or referring to (33), the nonlinear term may indeed be neglected in the limit $y \to \infty$. This linear far field is treated by Salathé (1969) and Chong & Sirovich (1970).

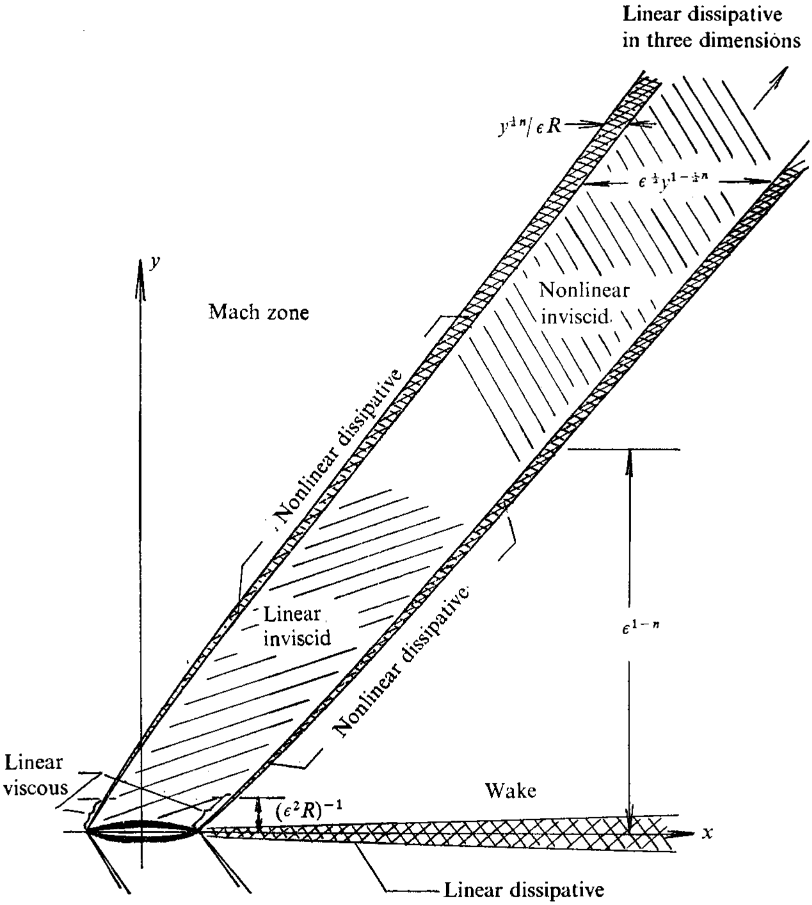

Figure 1. Various regions for supersonic flow past a body.

Figure 1 summarizes the theories which apply in the various portions of the flow field around a thin body of thickness ratio ϵ. Only the field in the upper half of some meridian plane in three dimensions or the upper half-plane in two dimensions is shown. Dissipative regions are designated by cross-hatching and the inviscid regions are simply hatched. The boundary layer is not sketched. The line of demarcation between the linear and nonlinear inviscid regions is at $1/\epsilon$ and $1/\epsilon^2$ for two and three dimensions, respectively, while the width of the inviscid wave zone grows as $O(\epsilon^{\frac{1}{2}}y^{\frac{1}{2}})$ and $O(\epsilon^{\frac{1}{4}}y^{\frac{1}{4}})$ in two and three dimensions, respectively. The extent of the linear viscous shock zone by (22) and (23) is simply of $O((\epsilon^2 R)^{-1})$. Thereafter the shock zone becomes nonlinear and dissipative.

To estimate nonlinear shock thickness we return to (22), which expresses the width of a weak shock in terms of its strength. From table 1 the strength is proportional to $\epsilon y^{-\frac{1}{2}n}$, and on substituting this into (22) we find that the thickness is given by

$$\delta/L \approx y^{\frac{1}{2}n}/\epsilon R. \tag{36}$$

The shock thickness in two dimensions grows at the same rate as the width of the wave zone, but in three dimensions the shock thickness ultimately completely engulfs the wave zone.

The above discussion indicates that for two dimensions the nonlinear inviscid wave zone, of width of $O(\epsilon^{\frac{1}{2}}y^{\frac{1}{2}})$, is enclosed by two much narrower shock regions of width of $O(y^{\frac{1}{2}}/\epsilon R)$. In I an explicit representation is found for the flow field past an arbitrary body and with some noteworthy exceptions the qualitative description given above is correct. Specifically at extreme distances, dissipative effects totally invade the flow. To see this we consider, for simplicity, flow past a thin diamond-shaped profile. Then from I we have in the limit $y \to \infty$, $y > R$, the expression

$$u \sim \frac{\epsilon U}{(M^2-1)^{\frac{1}{2}}} \frac{(2\tau/\epsilon^2k^2Ry)\exp\{-R[(\tau^2/2y)-\frac{1}{2}\epsilon k]\}}{\left(\frac{\pi y}{2R}\right)^{\frac{1}{2}}\left\{\operatorname{erfc}\left[(\frac{1}{2}+\tau)\left(\frac{R}{2y}\right)^{\frac{1}{2}}\right]+\operatorname{erfc}\left[(\frac{1}{2}-\tau)\left(\frac{R}{2y}\right)^{\frac{1}{2}}\right]\right\}+\frac{2}{\epsilon kR}\exp\left[-R\left(\frac{\tau^2}{2y}-\frac{\epsilon k}{2}\right)\right]}. \tag{37}$$

Over most of the range of variation of y the first term of the denominator of (37) is small and the Reynolds-number dependence is lost. This is precisely the linear profile predicted by the N-wave solution. But for y/R sufficiently large, the first term of the denominator is no longer negligible and in fact can dominate. From this it follows that the solution is everywhere structured by viscosity. The distances at which this occurrs are so extreme as to preclude any practical significance. It does caution us in regarding the flow as having a clearly separated dissipative shock zone.

This idea is more important in the three-dimensional case since our estimates show that the dissipative zone eventually occupies completely the disturbance region. This leads to the idea of a critical distance beyond which inviscid and dissipative regions cannot be separated. If the inviscid estimate of the wave zone is denoted by d then certainly when $\delta/d = O(1)$ the two zones are completely intertwined. Hence as a criterion for the critical distance we take that distance at which

$$\delta/d = O(\epsilon). \tag{38}$$

Alternatively, inviscid theory can be regarded as giving the position of the shock wave. From this one may construct a viscous shock wave based on the shock strength and location given by inviscid theory. Once this shock thickness becomes sufficiently large, the inviscid estimate of shock strength becomes erroneous. This argument also leads to (38). If we denote the critical distance by Y and use $\epsilon^{\frac{1}{2}}y^{\frac{1}{4}}$ for d/L then (36) and (38) give

$$Y = O(\epsilon^5R^2).$$

For example, taking $\epsilon = 10^{-1}$ and $R = 10^5$ gives us 10^5 body lengths as the critical distance. It should also be noted that since the Reynolds number decreases with altitude the effect of dissipation is enhanced.

Lastly we also give an estimate for the onset of linear dissipative flow in three dimensions. Referring to (32) we wish to find at what distance the dissipative term is large compared with the convective term. Since the wave region grows as $\epsilon^{\frac{1}{2}}y^{\frac{1}{4}}$ we can estimate that $\partial/\partial\tau = O(\epsilon^{-\frac{1}{2}}y^{-\frac{1}{4}})$. Using this in (32) we obtain $y \gg \epsilon^{\frac{2}{3}}R$

as the criterion for the onset of the linear dissipative regime. Since an N-wave is linear in its interior the last estimate is only valid if it is greater than Y. Therefore the linear regime applies for $y \gg \max[\epsilon^{\frac{3}{2}}R, \epsilon^5 R^2]$.

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